MOISTURE MIGRATION AND STRATIFIED TEXTURE IN FREEZING SOILS

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A calculation model is proposed for the problem of moisture migration in freezing soils. The location, thickness, and number of the ice lenses are calculated.

The usual models [1, 2] of the process of moisture migration in freezing soils are based on two assumptions: 1) moisture migration takes place only in the unfrozen soil; and 2) soil moisture migrates to a continuously moving freezing front. Recent evidence suggests that these assumptions are not realistic. Thus, it has been experimentally established that the freezing front does not always coincide with the ice-forming front but usually outstrips it, i.e., moisture migration takes place not only in the unfrozen but also in a certain zone of freezing soil [3, 4].

The freezing front is the boundary at which the principal part of the soil moisture is converted into ice at the temperature at which freezing begins. (We arbitrarily assume that this temperature is equal to 0°.) The ice-forming front is the boundary at which ice lenses form and grow. Obviously, the processes of moisture migration in the freezing soil and the underlying unfrozen soil are continuously related, and therefore in the calculation model itself it is necessary to take into account this continuous relationship between the mass transfer processes in the unfrozen and freezing zones.

Soil moisture migrates to the stationary ice-forming front, i.e., to the growing lenses and interlayers of ice. In calculating the process of moisture migration to the growing ice lenses, it is generally necessary to consider the triple system "ice–freezing soil–unfrozen soil." This system more closely reflects the essence of the crystallization-film mechanism of moisture migration, since the moisture migrates to the stationary ice-forming front and not to the moving freezing front. The processes of formation of the different lenses and layers of ice are essentially the same and differ only quantitatively; therefore it is sufficient to consider in general form the process of formation of a single lens.

The expression describing the motion of the freezing front is, in general, a continuous function of time. During the motion of the freezing front, of course, there may be pauses when the flow of heat to the freezing front is equal to the flow of heat away from it. At the same time, the motion of the ice-forming front is discontinuous in character: a sequence of pauses at the boundary of ice lens formation and growth with sudden shifts to a new (lower-lying) boundary. The duration of the pause is determined by the time of growth of the ice lens at the expense of the moisture of the unfrozen soil. At the moment this growth ceases, the ice-forming front descends abruptly to a new level, where the process of lens formation and growth recommences. Obviously, a pause in the motion of the freezing front is possible only when it coincides with the ice-forming front. It should be emphasized that the migration of moisture to the stationary ice-forming front takes place both during a certain pause and during the subsequent motion of the freezing front. Thus, at the boundary of the moving freezing front we specify not a certain value of the moisture potential (as in existing models), but equality of the potentials and migrating moisture fluxes in the unfrozen and freezing soils.

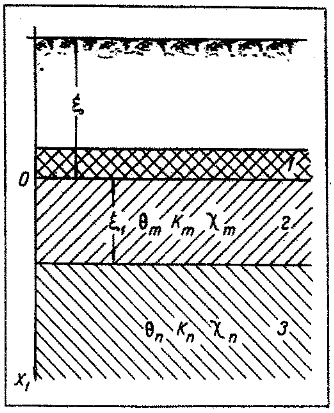

Fig. 1. Calculation model of the growth of an ice lens.

This treatment makes it possible to determine the cryogenic texture of the frozen soil, i.e., to calculate the thickness of the ice lenses and the mineral interlayers over the entire depth of the active layer.

The calculation model of the moisture migration process is shown schematically in Fig. 1. In calculating the growth of each ice lens we place the coordinate origin at the corresponding ice-forming front. The following notation has been employed: θ is the moisture potential; K is the potential diffusivity; χ is the mass conductivity; ξ is the freezing depth; τ is time; x is a coordinate.

The parameters x_1, ξ_1, and τ_1 relate only to the given ice lens; the parameters without a subscript 1, i.e., x, ξ, and τ, relate to the entire freezing layer. The boundary $x_1 = 0$ (at depth ξ) represents the lower face of the growing ice lens. Below the ice lens lie the freezing zone ($0 \le x_1 \le \xi_1$) and the unfrozen soil ($\xi_1 \le x_1 \le \infty$). The parameters θ, K, and χ relating to the freezing zone, are denoted by the subscript m, those relating to the unfrozen zone by the subscript n. In Fig. 1, the ice lens, the freezing zone, and the unfrozen soil are numbered 1, 2, and 3, respectively.

At the surface $x_1 = 0$, i.e., at the ice-forming front, we assign a moisture potential which should be regarded as the energy characteristic of the moisture field of the soil in direct contact with the growing ice lens. Since the temperature at the ice-forming front varies with time, the value of the corresponding potential will generally be time varying. At the boundary $x_1 = \xi_1$, i.e., at the boundary between the freezing and unfrozen soils, as already noted, we assume equality of the potentials and moisture fluxes. The initial distribution of the moisture potential of the unfrozen soil beneath the growing ice lens will not be constant with respect to depth. However, to simplify the problem, we assume a constant potential, which in general leads to a certain increase in the thickness of the ice lenses.

Before proceeding to a mathematical formulation of the problem, we will make a number of auxiliary calculations to determine the lens growth time ($\tau_c + \tau_p$), the depth l to which the soil is frozen beneath the lens, and the temperature and mean value of the moisture potential at the ice-forming front, t_i, θ_{av}, respectively.

At the surface of the soil we take the mean winter value of the temperature t_B; the initial temperature of the soil is assumed constant with respect to depth and equal to t_0. Using the method of successive substitution of quasi-stationary states in the frozen zone, we write the Stefan condition at the freezing boundary

$$\frac{\lambda_f t_B}{\xi} - \sigma w \frac{d\xi}{d\tau} = \frac{\lambda_{unf} t_0}{\sqrt{\pi a_{unf}}} \times$$

$$\times \frac{1}{\sqrt{\tau}} \frac{\exp\left[-\left(\frac{\alpha}{2\sqrt{a_{unf}}}\right)^2\right]}{\operatorname{erfc}\left(\frac{\alpha}{2\sqrt{a_{unf}}}\right)} \cong$$

$$\cong \frac{\lambda_{unf} t_0}{\sqrt{\pi a_{unf}}} \frac{1}{\sqrt{\tau}} \left(1 + 1.33 \frac{\alpha}{2\sqrt{a_{unf}}}\right), \qquad (1)$$

whence we find the averaged law of freezing

$$\xi = \alpha \sqrt{\tau},$$

where

$$\alpha = \sqrt{\left(\frac{\lambda_{unf} t_0}{\sqrt{\pi a_{unf}} Q}\right)^2 + \frac{2\lambda_f t_B}{Q}} - \frac{\lambda_{unf} t_0}{\sqrt{\pi a_{unf}} Q};$$

$$Q = \sigma w + \frac{1.33 \lambda_{unf} t_0}{\sqrt{\pi a_{unf}}}; \qquad (2)$$

λ_f and λ_{unf} are the thermal conductivities of the frozen and unfrozen soil; a_{unf} is the thermal diffusivity of the unfrozen soil; w is the volume moisture content of the soil; and σ is the specific heat of phase transitions. In what follows, for simplicity, we assume that $t_0 = 0$.

One of the principal problems is the determination of the lens growth time (in other words, the duration of the pause in the motion of the ice-forming front). We arbitrarily assume that the formation and growth of an ice lens begins only after the overlying lens ceases to grow at the expense of moisture from the unfrozen soil. The lens growth time is composed of the time during which the freezing front pauses (τ_p) and the time during which it is in motion (τ_c) up to the next pause. As the lens grows (after disturbance of thermal equilibrium at the ice-forming front), its temperature falls, since the freezing front is sinking. As the temperature of the freezing soil falls, the amount of unfrozen moisture decreases. A moment arrives when the temperature of the freezing soil at the ice-forming front becomes equal to the temperature T_c at which the migration of moisture from the unfrozen soil through the zone of freezing soil to the growing ice lens ceases. Thus, in the first approximation we can define the lens growth time (τ_c) as the time taken by the soil temperature at the ice-forming front to pass from the temperature at which freezing begins to the temperature T_c. This time is given by the expression

$$\tau_c = \frac{1}{\alpha^2}\left[\left(\frac{t_B}{t_B - T_c}\right)^2 - 1\right]\xi^2. \qquad (3)$$

The freezing depth of the soil beneath the ice lens (in other words, the thickness of the mineral interlayer) is

$$l = F\xi, \qquad (4)$$

where $F = T_c/(t_B - T_c)$.

The law of freezing of the soil beneath the lens can be represented as

$$\xi_1 = \alpha\left(\sqrt{\tau + \tau_0} - \sqrt{\tau_0}\right),$$

where τ_0 is the freezing time up to formation of the ice lens in question at depth ξ.

To simplify the problem, we will take a simpler law of motion of the freezing front beneath the growing lens:

$$\xi_1 = \alpha_1 \sqrt{\tau_1},$$

where $\alpha_1 = T_c(t_B - T_c)\alpha/t_B\,(T_c(2t_B - T_c))^{1/2}$. We determine α_1 from the condition that at time τ_c the thickness of the frozen soil will be equal to l. The temperature of the soil at the ice-forming front (at depth ξ) varies with time according to the law

$$t_i = \frac{t_B}{\xi} \alpha_1 \sqrt{\tau_1}.$$

Assuming a linear relation between the moisture potential and the temperature of the freezing soil, we write the expression for the moisture potential at the ice-forming front:

$$\theta_m|_{x_1=0} = \theta_1 + n\sqrt{\tau_1},$$

where θ_1 is the value of the potential at the temperature at which freezing begins; n is a proportionality factor.

Furthermore, we determine the mean integral value of the moisture potential at the ice-forming front in the presence of a moving freezing front:

$$\theta_{av} = \frac{1}{\tau_c}\int_0^{\tau_c} n\sqrt{\tau_1}\, d\tau_1.$$

The mathematical formulation of the problem of moisture migration to an ice-forming front in the presence of a moving freezing front has the form

$$\frac{\partial\theta_m}{\partial\tau_1}=K_m\frac{\partial^2\theta_m}{\partial x_1^2},\quad 0\leqslant x_1\leqslant\xi_1, \tag{5}$$

$$\frac{\partial\theta_n}{\partial\tau_1}=K_n\frac{\partial^2\theta_n}{\partial x_1^2},\quad \xi_1\leqslant x_1\leqslant\infty, \tag{6}$$

$$\theta_m=\theta_1+n\sqrt{\tau_1}\quad \text{at}\quad x_1=0, \tag{7}$$

$$\theta_n=\theta_0\quad \text{at}\quad x_1=\infty, \tag{8}$$

$$\theta_m=\theta_n\quad \text{at}\quad x_1=\xi_1, \tag{9}$$

$$\chi_m\frac{\partial\theta_m}{\partial x_1}=\chi_n\frac{\partial\theta_n}{\partial x_1}\quad \text{at}\quad x_1=\xi_1,$$
$$\xi_1=\alpha_1\sqrt{\tau_1}. \tag{10}$$

Our solution of problem (5)–(10) has the form

$$\theta_m=\theta_1+\frac{\theta_0-\theta_1}{\operatorname{erf}\left(\frac{\alpha_1}{2\sqrt{K_m}}\right)+M\operatorname{erfc}\left(\frac{\alpha_1}{2\sqrt{K_n}}\right)}\times$$
$$\times\operatorname{erf}\left(\frac{x_1}{2\sqrt{K_m\tau_1}}\right)+\frac{1}{\alpha_1}\left[C_1\operatorname{ierfc}\left(\frac{\alpha_1}{2\sqrt{K_n}}\right)-\right.$$
$$\left.-\sqrt{\pi}\,n\operatorname{ierfc}\left(\frac{\alpha_1}{2\sqrt{K_n}}\right)\right]x_1+$$
$$+\sqrt{\pi\tau_1}\,n\operatorname{ierfc}\left(\frac{x_1}{2\sqrt{K_m\tau_1}}\right), \tag{11}$$

$$\theta_n=\theta_0-\frac{\theta_0-\theta_1}{\operatorname{erf}\left(\frac{\alpha_1}{2\sqrt{K_m}}\right)+M\operatorname{erfc}\left(\frac{\alpha_1}{2\sqrt{K_n}}\right)}\times$$
$$\times M\operatorname{erfc}\left(\frac{x_1}{2\sqrt{K_n\tau_1}}\right)+$$
$$+C_1\sqrt{\tau_1}\operatorname{ierfc}\left(\frac{x_1}{2\sqrt{K_n\tau_1}}\right), \tag{12}$$

where

$$M=\frac{\chi_m}{\chi_n}\sqrt{\frac{K_n}{K_m}}\exp\left[\frac{\alpha_1^2}{4}\left(\frac{1}{K_n}-\frac{1}{K_m}\right)\right];$$

$$C_1=\left[\sqrt{\pi}\,n\left[\operatorname{ierfc}\left(\frac{\alpha_1}{2\sqrt{K_m}}\right)\frac{\chi_m}{\alpha_1}+\right.\right.$$
$$\left.\left.+\operatorname{erfc}\left(\frac{\alpha_1}{2\sqrt{K_m}}\right)\frac{\chi_m}{2\sqrt{K_m}}\right]\right]\left[\operatorname{ierfc}\left(\frac{\alpha_1}{2\sqrt{K_n}}\right)\times\right.$$
$$\left.\times\frac{\chi_m}{\alpha_1}+\operatorname{erfc}\left(\frac{\alpha_1}{2\sqrt{K_n}}\right)\frac{\chi_n}{2\sqrt{K_n}}\right]^{-1}.$$

We determine the moisture flow to the ice-forming front in the presence of a moving freezing front:

$$j=-\chi_m\left.\frac{\partial\theta_m}{\partial x_1}\right|_{x_1=0}=$$
$$=-\chi_m\left\{\frac{\theta_0-\theta_1}{\operatorname{erf}\left(\frac{\alpha_1}{2\sqrt{K_m}}\right)+M\operatorname{erfc}\left(\frac{\alpha_1}{2\sqrt{K_n}}\right)}\times\right.$$
$$\times\frac{1}{\sqrt{\pi K_m\tau_1}}+\frac{1}{\alpha_1}\left[C_1\operatorname{ierfc}\left(\frac{\alpha_1}{2\sqrt{K_n}}\right)-\right.$$
$$\left.\left.-\sqrt{\pi}\,n\operatorname{ierfc}\left(\frac{\alpha_1}{2\sqrt{K_m}}\right)\right]-\frac{\sqrt{\pi}\,n}{2\sqrt{K_m}}\right\}. \tag{13}$$

If in problem (5)–(10) instead of condition (6) we take the boundary condition

$$\theta_m=\theta_{av}\quad \text{at}\quad x_1=0, \tag{14}$$

the solution will have the simpler form

$$\theta_m=\theta_{av}+\frac{\theta_{av}-\theta_0}{D}\operatorname{erf}\left(\frac{x_1}{2\sqrt{K_m\tau_1}}\right), \tag{15}$$

$$\theta_n=\theta_0-\frac{\theta_{av}-\theta_0}{D}C_2\operatorname{erfc}\left(\frac{x_1}{2\sqrt{K_n\tau_1}}\right), \tag{16}$$

$$C_2=\frac{\chi_m}{\chi_n}\sqrt{\frac{K_n}{K_m}}\times$$
$$\times\exp\left[-\left(\frac{\alpha_1}{2\sqrt{K_m}}\right)^2+\left(\frac{\alpha_1}{2\sqrt{K_n}}\right)^2\right];$$

$$D=C_2\left[\operatorname{erf}\left(\frac{\alpha_1}{2\sqrt{K_n}}\right)-1\right]-\operatorname{erf}\left(\frac{\alpha_1}{2\sqrt{K_m}}\right),$$

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx,$$

$$\operatorname{erfc}(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-x^2}dx,$$

where

$$\operatorname{ierfc}(x)=\int_x^\infty\operatorname{erfc}(\xi)\,d\xi.$$

We will determine the flow of moisture to a growing ice lens in the presence of a moving freezing front:

$$i=-\chi_m\left.\frac{\partial\theta_m}{\partial x_1}\right|_{x_1=0}=\frac{A_1}{\sqrt{\tau_1}}, \tag{17}$$

where $A_1=\chi_m((\theta_{av}-\theta_0)/D)\,(1/(\pi K_m)^{1/2})$. For simplicity, in the subsequent calculations we will use expression (17) instead of the more complete (13).

Above we noted that the growth of the lens proceeds both during a certain pause in the motion of the freezing front and during its subsequent continued descent. During this pause (τ_p) the freezing front and the ice-forming front are at the same boundary. To determine τ_p we analyze the expression for the total heat flow q to the freezing front during a pause:

$$q=\sigma A_1/\sqrt{\tau_1}. \tag{18}$$

It is one of the basic assumptions of any model of the moisture migration process that all the moisture attracted to the ice-forming front freezes rapidly. In other words, q should not exceed the flow of heat away from the freezing front, which is equal to $\lambda_f t_B/\xi$.

Since $\lim_{\tau_1\to 0} q(\tau_1)=\infty$, during τ_p the freezing front will coincide with the ice-forming front and moisture migration will take place only in the unfrozen soil. The mathematical formulation of the problem of moisture migration to the ice-forming front in the presence of a stationary freezing front has the form

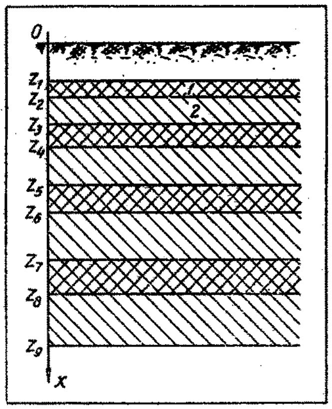

Fig. 2. Schematic representation of the texture of a frozen soil.

$$\frac{\partial \theta_n}{\partial \tau_1} = K_n \frac{\partial^2 \theta_n}{\partial x_1^2}, \quad 0 \leqslant x_1 \leqslant \infty, \tag{19}$$

$$\theta_n = \Theta_1 \quad \text{at} \quad x_1 = 0, \tag{20}$$

$$\theta_n = \Theta_0 \quad \text{at} \quad x_1 = \infty. \tag{21}$$

As a result of solving problem (19)–(21) we obtain the following expression for the migration flux:

$$i = A_2 / \sqrt{\tau_1}, \tag{22}$$

where $A_2 = -\chi_n(\theta_0 - \theta_1)/(\pi K_n)^{1/2}$.

The time τ_p is found from the condition

$$\frac{\lambda_f t_B}{\xi} = \frac{\sigma A_2}{\sqrt{\tau_p}}, \tag{23}$$

whence $\tau_p = (A_2\sigma/\lambda_f t_B)^2 \xi^2$.

The moisture accumulation during time τ_p is equal to

$$Q_p = \frac{2A_2^2 \sigma}{\lambda_f t_B} \xi. \tag{24}$$

The moisture accumulation during the motion of the freezing front is equal to

$$Q_m = \int_{\tau_2}^{c} \frac{A_1}{\sqrt{\tau_1}} d\tau_1 = 2A_1(\sqrt{\tau_c} - \sqrt{\tau_2}), \tag{25}$$

where τ_2, determined from the condition $\sigma A_1/(\tau_2)^{1/2} = \lambda_f t_B/\xi$, is given by

$$\tau_2 = \left(\frac{A_1 \sigma}{\lambda_f t_B}\right)^2 \xi^2.$$

Substituting the expression for τ_c and τ_2 into (25), we obtain

$$Q_m = 2A_1 \left[\sqrt{\frac{1}{\alpha^2}\left[\left(\frac{t_B}{t_B - T_c}\right)^2 - 1\right]} - \frac{A_1 \sigma}{\lambda_f t_B}\right] \xi. \tag{26}$$

Adding expressions (24) and (26) and multiplying the sum obtained by the coefficient 1.09, we find the thickness of the ice lens (r) for the entire period of its growth:

$$r = R\xi, \tag{27}$$

where

$$R = 1.09 \left\{\frac{A_2^2 \sigma}{\lambda_f t_B} + 2A_1 \left[\sqrt{\frac{1}{\alpha^2}\left[\left(\frac{t_B}{t_B - T_c}\right)^2 - 1\right]} - \frac{A_1 \sigma}{\lambda_f t_B}\right]\right\}.$$

Thus, we have determined the thickness of the ice lens and the underlying mineral layer:

$$r = R\xi; \; l = F\xi. \tag{28}$$

To construct the texture of the frozen soil it is necessary to know the depth of the first ice lens. The growth of ice layers begins at a freezing rate not exceeding the so-called critical freezing rate v_c. The value of v_c depends on the properties of the soil and the moisture content and is determined experimentally. Knowing v_c, we can find the depth (z_1) of the first lens:

$$z_1 = \alpha^2/2v_c. \tag{29}$$

Figure 2 is a schematic representation of the texture of a frozen soil. The ice lenses are denoted by the numeral 1, the mineral layers by the numeral 2. The depth of the first lens is given by expression (29). From (28) we successively determine the values z_2, z_3, z_4, ... :

$$z_2 = z_1(1+R); \; z_3 = z_1(1+R)(1+F);$$
$$z_4 = z_1(1+R)^2(1+F).$$

Hence the thicknesses of the ice lenses are, respectively, equal to

$$Rz_1; \; Rz_1(1+R)(1+F),$$
$$Rz_1(1+R)^2(1+F)^2, \; Rz_1(1+R)^3(1+F)^3,$$

or, in general form,

$$Rz_1(1+R)^{\mu-1}(1+F)^{\mu-1}, \quad \text{where} \quad \mu = 1, 2, 3, \ldots$$

The total thickness of all the lenses is equal to the heave of the soil (h_s) and is given by the expression

$$h_s = \sum_{\mu=1}^{\mu=s} Rz_1(1+R)^{\mu-1}(1+F)^{\mu-1},$$

where s is the number of lenses.

To determine s, we assume, for example, that the frozen layer of soil ends in a mineral interlayer. The coordinates of the lower surfaces of the mineral interlayers ($z_3, z_5, z_7, \ldots$) are, respectively, equal to

$$z_3 = z_1(1+R)(1+F),$$
$$z_5 = z_1(1+R)^2(1+F)^2, \; z_7 = z_1(1+R)^3(1+F)^3,$$

or, in general form,

$$z_1(1+R)^{\mu}(1+F)^{\mu}, \text{ where } \mu = 0,\ 1,\ 2,\ 3,\ \ldots$$

The coordinate of the lowest mineral interlayer (μ = s) is equal to

$$z_1(1+R)^s(1+F)^s.$$

The number of ice lenses is found from the condition

$$z_1(1+R)^s(1+F)^s = h_g,$$

where h_g is the thickness of the frozen layer of soil. Hence,

$$s = \frac{\ln \frac{h_g}{z_1}}{\ln[(1+R)(1+F)]}.$$

Thus, from the formulas presented above it is possible to determine the texture of a frozen soil.

In conclusion we note the following. The obtained linear relation between the thickness r of the ice lenses, the mineral interlayers l and the freezing depth ξ is attributable to the introduction of a constant soil temperature t_B at the surface and a simplified law of freezing of the soil beneath the lens ($\xi_1 = \alpha_1(\tau_1))^{1/2}$). Taking into account the variation of soil temperature with time and the true law of freezing leads to serious mathematical difficulties and, moreover, does not greatly increase the accuracy of the calculations, since the determination of the heat and mass transfer characteristics of soils is, in any case, approximate.

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